

Theorem (2): Prove that the identity element in a group is unique.

Proof: Let us suppose that G be a group and let e be the identity element.
we have to prove here that e is unique.

If possible, suppose e is not unique.

we suppose then e' be another identity element in a group G .

Since e is the identity of G ,

$$\text{therefore, } ae = ea = a \quad \text{--- (1)}$$

Again, since e' is the identity element of G ,

$$\therefore \text{ therefore, } ae' = e'a = a \quad \text{--- (2)}$$

Since the equation (1) is true $\forall a \in G$ and since $e' \in G$

Therefore, putting $a = e'$ in (1), we get

$$e'e = ee' = e' \quad \text{--- (3)}$$

Similarly putting $a = e$ in (2), we get

$$ee' = e'e = e \quad \text{--- (4)}$$

Hence from (3) & (4), it follows obviously that

$e = e'$, which means that e' is different from e .

hence our supposition is false, and the theorem is true.

i.e. identity in a group is unique.

Hence ~~the~~ proved.

Theorem (3) Prove that the inverse of an element in a group is unique.

Proof: Suppose G be a group. let $a \in G$ and a^{-1} be the inverse of a .

We have now to prove that a^{-1} is unique. if possible suppose a^{-1} is not unique and let \bar{a}^{-1} is another inverse of a ; therefore,

$$aa^{-1} = a\bar{a}^{-1} = e \quad \text{--- (1)}$$

Similarly, since a^{-1} is the inverse of a , therefore,

$$a\bar{a}^{-1} = a^{-1}a = e \quad \text{--- (2)}$$

where e is the identity element of G .

Multiplying (1) by a^{-1} on the left, we get

$$a^{-1}(aa^{-1}) = a^{-1}e = a^{-1} \quad \text{--- (3)}$$

Multiplying (2) by a^{-1} on the right, we get

$$(a^{-1}a)a^{-1} = ea^{-1} = a^{-1} \quad \text{--- (4)}$$

But we know by associative law

$$a^{-1}(aa^{-1}) = (a^{-1}a)a^{-1}$$

Therefore, we have from (3) & (4)

$$a^{-1} = a^{-1}$$

Hence our supposition is false and the ~~inverse~~ inverse of an element in a group is unique. Hence the theorem proved.